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Statistical Mechanical Theory of the Great Red Spot of Jupiter

Julien Michel[1] **and Raoul Robert**[1]

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In a previous study a permanent isolated vortex like the Great Red Spot of Jupiter was obtained as a statistical equilibrium for the classical quasi-geostrophic model of atmospheric motion on rapidly rotating planets. We provide here a theoretical basis for this work and relate it to a previous model of the spot (Rossby soliton).

KEY WORDS: Quasigeostrophic atmospheric motion; Jovian planets; statistical equilibrium states.

1. INTRODUCTION

The Great Red Spot of Jupiter is a huge atmospheric vortex which has persisted for at least three centuries in the shear between two zonal jets. The persistence of this structure in a highly turbulent environment seems mysterious.

We shall use the classical quasigeostrophic model (given here in Section 2) to describe the atmospheric dynamics on rapidly rotating planets. This model results from great simplifications and is rather crude, but we believe that it retains the main mechanisms leading to the remarkable features of the atmospheric dynamics of the outer planets. We refer to Ingersol[(9)] for a clear and lucid survey on this point.

One can readily see that the spot cannot be explained by a classical linearization of the quasigeostrophic equations (leading to Rossby waves). Indeed, there is no Rossby wave reproducing the main features of the spot, and, since these waves are dispersive, an isolated structure formed of such

[1] CNRS, Laboratoire d'Analyse Numérique, Université Lyon 1, F-69622 Villeurbanne Cedex, France.

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waves must desintegrate. Thus it was viewed as a nonlinear phenomenon and a description in terms of solitary wave was proposed.[14, 22, 28]

In a recent paper Sommeria *et al.*[30] gave an explanation of the phenomenon on a statistical mechanical level, as the most probable state in an atmosphere with strong turbulent mixing, using a natural extension of a statistical equilibrium theory for 2D Euler equations.[25, 26] Actually we believe that the generation of small-scale vorticity structures (by hydrodynamic instabilities) and then the convergence toward a statistical equilibrium is the key mechanism leading to structures like the Great Red Spot.

Our aim is here to give some clarification on the theoretical grounds needed in this approach. We show that the quasigeostrophic equations satisfy the conditions required to apply the theory of ref. 16. An alternative approach based on the classical point-vortices approximation is also studied.

Following ref. 30, we describe a structure like the spot as an equilibrium state for the quasigeostrophic model. We show that such an equilibrium state is a solitary wave; so our approach is consistent with previous ones. Let us notice here that the description of the spot as a solitary wave is in fact very vague since, as we shall see, there are many such nonlinear waves. Our approach predicts more precisely what sort of solitary wave it might be.

A closely related work, devoted to a detailed discussion of the physical assumptions and the comparison of the results from numerical computations with the observational data from space probes, will be published soon.[31]

2. THE QUASIGEOSTROPHIC (QG) MODEL FOR THE GREAT RED SPOT

Although Jupiter is probably fluid up to great depth, it is an acknowledged fact that the observed vortices are confined in a shallow gaseous layer, due to a stable density stratification. On the other hand, zonal jets probably exist at a greater depth. To describe this system we shall use the model given in refs. 9 and 10: a shallow layer of fluid of constant density ρ_1 and mean depth D is floating on a much thicker layer of constant density $\rho_2 > \rho_1$. We shall suppose that the atmospheric flow in the superficial layer is approximately two-dimensional and incompressible. We suppose also that the relative motion is slow compared to the rotation period of the planet $2\pi/\omega$, so that the classical quasigeostrophic approximation holds. We use also the beta-plane approximation: the motion takes place in a zonal band about a reference latitude θ_0; we assimilate this region to a cylinder by means of the coordinates x = longitude, $y = (\theta - \theta_0)/\cos\theta_0$.

Applying Kelvin's theorem then yields the material conservation of the potential vorticity (PV) q:

$$\frac{dq}{dt} = q_t + \mathbf{u} \cdot \nabla q = 0 \tag{I}$$

where $\mathbf{u}$ is the relative velocity field of the superficial motion, and q is given by

$$-\Delta\psi + \beta_0 y + c^2(\psi - \psi_d) = q \tag{II}$$

where ψ is the stream function of $\mathbf{u}$ and ψ_d the stream function of the deep flow (supposed given),

$$\beta_0 = \cos^2\theta_0/|\sin\theta_0|, \qquad c^2 = L^2/\mathscr{R}^2$$

with $L = R\cos\theta_0$ (R is the radius of the planet) and $\mathscr{R}$ is the Rossby deformation radius:

$$\mathscr{R}^2 = \frac{\rho_2 - \rho_1}{\rho_2}\frac{gD}{f_0^2}$$

$f_0 = 2\omega\sin\theta_0$ is the Coriolis parameter and g is the acceleration due to gravity.

Remarks. 1. Equations (I) and (II) are dimensionless; time has been scaled by the Coriolis parameter (i.e., new time $t = |f_0|\, t'$).

2. The unknown functions $q(\mathbf{x})$, $\psi(\mathbf{x})$, $\mathbf{x} = (x, y)$, are supposed to be 2π-periodic in x, and the equations holds on $\mathbb{R} \times \,]-m, m[$. In the sequel we shall denote $\mathscr{C} = (\mathbb{R}/2\pi\mathbb{Z}) \times \,]-m, m[$, the cylinder, and $\Omega = \,]0, 2\pi[\times \,]-m, m[$.

3. In order that (I) and (II) define a dynamical system, ψ has to be recovered from q; thus we must make precise the boundary conditions satisfied by ψ.

It comes from the geostrophic equilibrium[10, 23] that $\psi = -(g/L^2 f_0\, |f_0|)\, \eta$, with η being the small deformation of the shallow layer surface. Thus the mass conservation implies that $\int_\Omega \psi\, d\mathbf{x} = 0$. Now, since the flow is confined in the zonal band, we shall assume that $\mathbf{u}$ is tangent to the boundaries $y = \pm m$, so that ψ is constant $= \psi_+$ (resp. ψ_-) on $y = +m$ (resp. $-m$).

But $2\pi(\psi_+ - \psi_-)$ is equal to the first component of the impulsion vector (which is conserved, due to the invariance of the domain by the translations along the x axis). We shall suppose in what follows that the impulsion is zero in the rotating frame of reference, so that $\psi_+ = \psi_-$, and this boundary value can be calculated from the condition $\int_\Omega \psi \, d\mathbf{x} = 0$.

4. We shall suppose also in our model that the deep flow is steady and x-wise directed. Then the conjugate influence of beta effect (variation of the Coriolis parameter with the latitude) and deep flow can be resumed by introducing the topography function

$$h(y) = c^2 \psi_d(y) - \beta_0 y$$

Finally, replacing ψ by $\psi - \psi_+$, one gets the dynamical system:

$$(\mathrm{QG}) \begin{cases} q_t + \operatorname{curl} \psi \cdot \nabla q = 0 & (\mathrm{QG1}) \\ \left.\begin{cases} -\Delta\psi + c^2(\psi - \bar{\psi}) = h(y) + q \\ \psi = 0 \quad \text{for} \quad y = \pm m, \quad 2\pi\text{-periodic in } x \end{cases}\right\} & (\mathrm{QG2}) \end{cases}$$

where we denote $\bar{\psi} = (1/|\Omega|) \int_\Omega \psi \, d\mathbf{x}$.

3. MAIN PROPERTIES OF THE SYSTEM (QG)

Notice first that (QG), which appears as a transport equation (QG1) coupled with the elliptic problem (QG2), is very similar to the Euler system which governs the motion of a 2D incompressible perfect fluid (in the velocity–vorticity formulation). Indeed, the Euler system is obtained for $c^2 = 0$ and $h = 0$, in which case q is equal to the usual vorticity.

Like the Euler system, (QG) can be put in a Hamiltonian form (at least formally).[20] Indeed, we can write (QG1)

$$q_t = J(q)[\nabla H(q)]$$

where $H(q) = \frac{1}{2} \int_\Omega \psi(q + h) \, d\mathbf{x}$ [ψ is associated to q by (QG2)] is the total energy of the system.

For a variation δq (and corresponding variation $\delta\psi$ for ψ), the first variation δH of H is

$$\delta H = \frac{1}{2} \int_\Omega \delta\psi(q + h) \, d\mathbf{x} + \frac{1}{2} \int_\Omega \psi \delta q \, d\mathbf{x}$$

Integrating by parts the first term, we get

$$\delta H = \int_\Omega \psi \delta q \, d\mathbf{x}$$

Thus $\nabla H(q)$ is identified with the stream function ψ. Here $J(q)$ is the skew-symetric operator defined by

$$J(q)[\varphi] = -\text{curl}\ \varphi \cdot \nabla q$$

The associated Poisson brackets are the same as for Euler equations;[20] the only change is in the Hamiltonian H.

3.1. Constants of the Motion

As for Euler equations, we have the family of Casimir invariants associated to the degenerate symplectic structure. These are the functionals

$$C_f(q) = \int_\Omega f(q(\mathbf{x}))\, d\mathbf{x}$$

for any continuous function f on $\mathbb{R}$. Let us define the distribution measure of q, π_q, by $\langle \pi_q, f \rangle = C_f(q)$. Then π_q is conserved by the flow.

Furthermore, the Hamiltonian H is conserved, and due to the particular geometry of the domain (cylinder), we have the supplementary invariant:

$$M = \int_\Omega yq\, d\mathbf{x}$$

Indeed, we have

$$\begin{aligned}
\frac{dM}{dt} &= \int_\Omega yq_t\, d\mathbf{x} = -\int_\Omega y\, \text{div}(q\mathbf{u})\, d\mathbf{x} \\
&= -\int_\Omega q\psi_x\, d\mathbf{x} \qquad \text{(integrate by parts)} \\
&= -\int_\Omega (-\Delta\psi + c^2(\psi - \bar{\psi}) - h)\, \psi_x\, d\mathbf{x} = \int_\Omega \Delta\psi\, \psi_x\, d\mathbf{x} \\
&= -\int_\Omega \nabla\psi \cdot \nabla\psi_x\, d\mathbf{x} = -\frac{1}{2}\int_\Omega ((\nabla\psi)^2)_x\, d\mathbf{x} = 0
\end{aligned}$$

3.2. Existence-Uniqueness Result for the Cauchy Problem

It can be shown that Yudovitch's arguments for Euler equations work as well for (QG) and give the existence of a unique weak solution for any initial datum q_0 in the phase space $L^\infty(\Omega)$.[15] We shall denote Φ_t: $L^\infty(\Omega) \to L^\infty(\Omega)$ the corresponding flow.

3.3. The Singular QG Model (SQG)

Onsager's statistical view of hydrodynamics was based on an analogy with the N-point vortices model.[21] For the QG model also we can approximate the potential vorticity by a finite linear combination of Dirac masses and derive (at least formally) a finite-dimensional Hamiltonian system giving the dynamics of the cloud of vortices.

Taking f for the right-hand side in (QG2), we denote by ψ_f the corresponding solution; we have

$$\psi_f(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\mathbf{x}'$$

where $G(\mathbf{x}, \mathbf{x}')$ is the Green function of the elliptic problem (QG2).

As for the Laplacian, we have

$$G(\mathbf{x}, \mathbf{x}') = -\frac{1}{2\pi} \operatorname{Log} |\mathbf{x} - \mathbf{x}'| + F(\mathbf{x}, \mathbf{x}')$$

where $F(\mathbf{x}, \mathbf{x}')$ is a symetric smooth function on $\Omega \times \Omega$.

For $f = h + \sum_{j=1}^N q_j \delta_{\mathbf{x}_j}$, we get

$$\psi_f(\mathbf{x}) = \psi_h(\mathbf{x}) + \sum_j q_j G(\mathbf{x}, \mathbf{x}_j)$$

Now, it follows from classical regularity results for Dirichlet's problem that ψ_h is C^1 on the cylinder $\mathscr{C}$. Indeed, since h is in L^∞, ψ_h is in the classical Sobolev space $H_0^1 \cap H^2$. By Sobolev's embedding theorem, ψ_h is continuous up to the boundary, and $-\Delta\psi_h = h - c^2(\psi_h - \bar{\psi}_h)$ is in L^∞. Thus $\mathbf{u}_h = \operatorname{curl} \psi_h$ satisfies a quasi-Lipschitz estimate:

$$|\mathbf{u}_h(\mathbf{x}) - \mathbf{u}_h(\mathbf{x}')| \leqslant c\, |\mathbf{x} - \mathbf{x}'|\, (1 + |\operatorname{Log} |\mathbf{x} - \mathbf{x}'||)$$

Then, removing the singular part with zero mean in the velocity field

$$\mathbf{u} = \mathbf{u}_h + \sum_j q_j \operatorname{curl} G(\mathbf{x}, \mathbf{x}_j)$$

we can assign a well-defined velocity to each point vortex $\mathbf{x}_i$. Hence we get (in a formal way) the finite-dimensional Hamiltonian system for the vortices. We have

$$\text{(SQG)} \qquad \left\{ \begin{array}{l} \dfrac{dx_i}{dt} = \dfrac{1}{q_i} \dfrac{\partial \mathscr{H}}{\partial y_i} (\mathbf{x}_1, \ldots, \mathbf{x}_N) \\ \dfrac{dy_i}{dt} = -\dfrac{1}{q_i} \dfrac{\partial \mathscr{H}}{\partial x_i} (\mathbf{x}_1, \ldots, \mathbf{x}_N) \\ \qquad i = 1, \ldots, N \end{array} \right.$$

where $\mathbf{x}_i = (x_i, y_i)$, $\mathscr{H}(\mathbf{x}_1, ..., \mathbf{x}_N) = \frac{1}{2} \sum_{k \neq j} q_k q_j G(\mathbf{x}_k, \mathbf{x}_j) + \frac{1}{2} \sum_k q_k^2 \mathscr{F}(\mathbf{x}_k) + \sum_k q_k \psi_h(\mathbf{x}_k)$ and $\mathscr{F}(\mathbf{x}) = F(\mathbf{x}, \mathbf{x})$.

We can prove that for any initial data of N distinct point vortices $\mathbf{x}_1(0), ..., \mathbf{x}_N(0)$ in Ω, the system SQG has a unique C^1 solution during a finite time (depending on the initial data). This is due to the fact that $\mathscr{F}(\mathbf{x})$ is a smooth function and $\mathbf{u}_h$ is quasi-Lipschitz.[34] Notice that, in general, nothing prevents the blowup of the system at a finite time (collision of vortices may occur).

Since (SQG) is Hamiltonian, the associated flow conserves the Liouville measure $d\mathbf{x}_1 \cdots d\mathbf{x}_N$ on the phase space Ω^N.

4. STATISTICAL EQUILIBRIUM STATES FOR THE QG MODEL

We define now equilibrium states for QG equations. We give two different approaches. The first one is based on the general Young measure framework given in ref. 16, while the second, which works only in the particular case of a potential vorticity taking a finite number of distinct values, uses an approximation of QG by a cloud of point-vortices (SQG).

4.1. The Young Measure Approach

To apply this method, we only have to check that (QG) belongs to the class defined in ref. 16. That is, the corresponding flow Φ_t on the phase space $L^\infty(\Omega)$ satisfies the hypotheses H1, H2, H3 of ref. 16 (since we shall not use them explicitly here, we do not recall these rather technical continuity assumptions on the flow). We refer to ref. 15 for a proof of this point; and we refer to Appendix A for a definition of Young measures.

Following ref. 16, we get the equilibrium set as follows.

Let us consider an initial datum q_0 in $L^\infty(\Omega)$; denote $r = |q_0|_\infty$, $\pi_0 = (1/|\Omega|)\, \pi_{q_0}$, $\pi = d\mathbf{x} \otimes \pi_0$, $\mathscr{M}$ the space of Young measures on $\Omega \times [-r, r]$.

To any Young measure ν in $\mathscr{M}$ (which one can visualize as a potential vorticity function oscillating in the neighborhood of every point), we associate the mean potential vorticity

$$\bar{\nu}(\mathbf{x}) = \int z \, d\nu_{\mathbf{x}}(z)$$

We shall say that ν is a mixture of q_0 if it satisfies

$$\int_\Omega \nu_{\mathbf{x}} \, d\mathbf{x} = \pi_{q_0}$$

Now let us introduce the closed subset of $\mathcal{M}$:

$$\mathcal{E} = \{ \nu \in \mathcal{M} \mid \nu \text{ mixture of } q_0,\ H(\bar{\nu}) = H(q_0),\ M(\bar{\nu}) = M(q_0) \}$$

According to ref. 16, we define the set $\mathcal{E}^*$ as the set of macrostates (Young measures) ν^* which are solutions of the variational problem

$$\text{(VP)} \qquad I_\pi(\nu^*) = \min_{\nu \in \mathcal{E}} I_\pi(\nu)$$

where $I_\pi(\nu)$ is the Kullback information functional, that is

$$I_\pi(\nu) = \int_{\Omega \times [-r, r]} \operatorname{Log} \frac{d\nu}{d\pi}\, d\nu \qquad \text{if } \nu \text{ is absolutely continuous with respect to } \pi$$

$$I_\pi(\nu) = +\infty \qquad \text{otherwise}$$

Notice here that $\mathcal{M}$ being compact and I_π a lower semicontinuous functional, (VP) always has a nonempty set of solutions $\mathcal{E}^*$.

Then the equilibrium set, corresponding to the initial datum q_0, is

$$\mathcal{E}_q^* = \{ \bar{\nu} \mid \nu \in \mathcal{E}^* \}$$

We recall that $\Phi_t \mathcal{E}_q^* = \mathcal{E}_q^*$.

4.2. The Point-Vortices Approach

In what follows C_c denotes the space of continuous, compactly supported, real functions on the cylinder $\mathcal{C} = (\mathbb{R}/2\pi\mathbb{Z}) \times]-m, m[$, M_b the space of all bounded Radon measures on $\mathcal{C}$, and M_1 the subset of probability measures. On M_b we shall use the narrow (resp. vague) topology, which is the weak topology associated to the continuous and bounded (resp. compactly supported) functions on $\mathcal{C}$.

We carry out this second method in the case where the initial potential vorticity $q_0(\mathbf{x})$ takes only a finite number of distinct values $q_1, \ldots, q_n$.

We have

$$\pi_0 = \frac{1}{|\Omega|} \pi_{q_0} = \sum_{i=1}^{n} P_i \delta_{q_i}$$

where

$$P_i = \frac{|\Omega_i|}{|\Omega|}, \qquad \Omega_i = \{ \mathbf{x} \in \Omega \mid q_0(\mathbf{x}) = q_i \}$$

Then we approximate the system (QG) by a singular model composed of a cloud of N vortices of n different types in the following way:

- The vortices of type i are located at the points $\mathbf{x}_k^i$, $k=1,\dots,N_i$.
- The number N_i is proportional to the area of Ω_i: $N_i=[NP_i]$ (integer part).
- All vortices of type i have the same strength $\gamma_i=|\Omega_i|\, q_i/N_i$.

Notice that the number of point vortices per unit area is independent of the type. Of course, one may consider other choices, for example, when all the vortices have the same strength; and that will certainly yield different equilibrium states. Our choice is justified by the fact that it will allow us to take into account all the constants of the motion of the continuous system (the energy and the area of each vorticity patch).

We are interested in the distribution of each type of vortex on the physical space $\mathscr{C}$. Thus, we consider the empirical distributions

$$\rho_N^i=\frac{1}{N_i}\sum_k \delta_{\mathbf{x}_k^i}, \qquad i=1,\dots,n$$

The space $\mathscr{C}^N$ will be endowed with the Liouville probability measure $d\tilde{\mathbf{x}}^1\otimes\cdots\otimes d\tilde{\mathbf{x}}^N$, where $d\tilde{\mathbf{x}}^i=(1/|\Omega|^{N_i})\, d\mathbf{x}_1^i\cdots d\mathbf{x}_{N_i}^i$. Then we denote by μ_N the probability distribution of the random variable $\rho_N=(\rho_N^1,\dots,\rho_N^n)$ on the space M_1^n.

Now we prove the following.

Proposition 4.1. For $N\to\infty$, the sequence μ_N has a large-deviation property (see Appendix B) with constants N and rate function

$$L(\nu^1\cdots\nu^n)=\sum_{i=1}^{n} P_i I_{d\tilde{\mathbf{x}}}(\nu^i), \qquad d\tilde{\mathbf{x}}=\frac{d\mathbf{x}}{|\Omega|}$$

Proof. Take $E=M_b^n$ endowed with the weak topology associated to the duality $\langle(\nu^1\cdots\nu^n), (\varphi^1\cdots\varphi^n)\rangle=\sum\langle\nu^i,\varphi^i\rangle$, where $\varphi=(\varphi^1\cdots\varphi^n)\in C_c^n$. Thus the dual of E is $E'=C_c^n$, and we compute the Laplace transform:

$$\begin{aligned}\hat{\mu}_N(N\varphi)&=\prod_i\int_{\mathscr{C}^{N_i}}\exp(N\langle\rho_N^i,\varphi_i\rangle)\, d\tilde{\mathbf{x}}^i\\ &=\prod_i\prod_k\int_{\mathscr{C}}\exp\left[\frac{N}{N_i}\varphi_i(\mathbf{x}_k^i)\right]d\tilde{\mathbf{x}}_k^i\end{aligned}$$

Hence

$$\frac{1}{N} \operatorname{Log} \hat{\mu}_N(N\varphi) = \sum_i \frac{N_i}{N} \operatorname{Log} \left\{ \int_{\mathscr{C}} \exp\left[\frac{N}{N_i} \varphi_i(\mathbf{x})\right] d\tilde{\mathbf{x}} \right\}$$

When $N \to \infty$, this converges toward

$$F(\varphi_1 \cdots \varphi_n) = \sum_i P_i \operatorname{Log} \left\{ \int_{\mathscr{C}} \exp\left[\frac{1}{P_i} \varphi_i(\mathbf{x})\right] d\tilde{\mathbf{x}} \right\}$$
$$= \sum_i h_i(\varphi_i)$$

F is an everywhere finite convex functional on E'. It is continuous for the norm topology; therefore it is also lower semicontinuous for the weak topology $\sigma(E', E)$, and condition 1 of Baldi's theorem is fulfilled (see Appendix B). The compacity condition 2 is also obviously satisfied since the closure of M_1 is compact for the vague topology.

Now we compute the Young–Fenchel transform $L = F^*$ of F,

$$L(\nu^1 \cdots \nu^n) = \sup_{\varphi_1 \cdots \varphi_n} \left(\sum_i \langle \nu^i, \varphi_i \rangle - \sum_i h_i(\varphi_i) \right)$$
$$= \sum_i h_i^*(\nu^i)$$

But

$$h_i^*(\nu) = P_i I_{d\tilde{\mathbf{x}}}(\nu) \qquad \text{for} \quad \nu \in M_1$$
$$= +\infty \qquad \text{for} \quad \nu \notin M_1$$

This easily follows from ref. 33.

Thus

$$L(\nu^1 \cdots \nu^n) = \sum_i P_i I_{d\tilde{\mathbf{x}}}(\nu^i)$$

where $I_{d\tilde{\mathbf{x}}}$ is extended by $+\infty$ out of M_1.

Then we deduce from ref. 16, Lemma 3.2, that Condition 3 is satisfied since L is strictly convex on dom L and

$$(P_1 \varphi_1, \ldots, P_n \varphi_n) \in \partial L(\exp(\varphi_1)\, d\tilde{\mathbf{x}}, \ldots, \exp(\varphi_n)\, d\tilde{\mathbf{x}}) \qquad \text{for} \quad \varphi_i \in C_c$$

Thus Baldi's theorem applies and the proposition is proved. ■

We are now going to carry out the thermodynamic limit of the random variables ρ_N when they are submitted to some constraints. These constraints will be given by the constants of the motion of the continuous system (QG). At first sight, this may seem surprising, since these constraints do not correspond to the constants of the motion of the discrete system (SQG). But we must keep in mind that we are actually interested in the equilibrium states of the continuous system; thus we find it physically relevant to constrain ρ_N by the actual conserved quantities of the continuous system.

4.3. The Incompressibility Constraint

The discrete system (SQG) naturally expresses that the total vorticity of each type i is conserved, but not the fact that the area occupied by each type is conserved.

Let us suppose that (for large N) each ρ_N^i is approximated (for the vague topology) by a density $\rho^i(\mathbf{x})\,d\tilde{\mathbf{x}}$. Then we have for the singular potential vorticity

$$Q_N = \sum_i \gamma_i \sum_k \delta_{\mathbf{x}_k^i} \approx q(\mathbf{x})\,d\mathbf{x}$$

where

$$q(\mathbf{x}) = \sum_i q_i P_i \rho^i(\mathbf{x})$$

Let us denote $p_i(\mathbf{x}) = P_i \rho^i(\mathbf{x})$. In the continuous case, $p_i(\mathbf{x})$ is the local portion of area occupied by the level q_i and we must have $\sum_i p_i(\mathbf{x}) = 1$ for all $\mathbf{x}$. In the discrete case, this relation means that the total number of particles in any region is proportional to the area. Of course this condition (which we call the incompressibility constraint) is not conserved by (SQG).

Finally we shall impose that ρ_N approximately satisfies the incompressibility constraint:

$$\text{(INC)} \qquad \sum_i P_i \nu^i = d\tilde{\mathbf{x}}$$

Let us denote by $\mathscr{E}_0$ the closed subset of M_1^n defined by (INC). For $(\nu^1 \cdots \nu^n)$ in $\mathscr{E}_0$, each ν^i is absolutely continuous with respect to $d\tilde{\mathbf{x}}$: $\nu^i = \rho^i(\mathbf{x})\,d\tilde{\mathbf{x}}$, and we have

$$\sum_i P_i \rho^i(\mathbf{x}) = 1, \qquad \text{hence} \quad 0 \leqslant \rho^i(\mathbf{x}) \leqslant \frac{1}{P_i}$$

4.4. The Energy Constraint

To each $(\nu^1 \cdots \nu^n)$in M_1^n we associate the potential vorticity

$$Q = |\Omega| \sum_i q_i P_i \nu^i$$

The energy of Q is not defined in general; but for $(\nu^1 \cdots \nu^n)$ in $\mathscr{E}_0$ we have $Q = q(\mathbf{x})\, d\mathbf{x}$ with $q(\mathbf{x})$ in $L^\infty(\mathscr{C})$, whence the energy $H(Q) = H(q)$ is finite. Furthermore, H is continuous on $\mathscr{E}_0$ for the vague topology.

We can define also $M(Q) = \int_{\mathscr{C}} y\, dQ(\mathbf{x})$, and finally for a given $q_0(\mathbf{x})$, we denote

$$\mathscr{E} = \{(\nu^1 \cdots \nu^n) \in \mathscr{E}_0 \mid H(Q) = H(q_0), M(Q) = M(q_0)\}$$

Thus we are led to carry out the thermodynamic limit for the random variables ρ_N with the microcanonical constraint $\rho_N \in \mathscr{E}$. As it is shown in the Concentration Theorem 2.2 of ref. 16, this amounts to solving the variational problem

$$\text{minimize} \quad L(\nu^1 \cdots \nu^n) \quad \text{on} \quad \mathscr{E}$$

and this obviously amounts to minimizing the functional

$$\int_\Omega \sum_i p_i(\mathbf{x}) \operatorname{Log} p_i(\mathbf{x})\, d\mathbf{x}$$

under the constraints

$$\sum_i p_i(\mathbf{x}) = 1 \tag{$*$}$$

$$\int_\Omega p_i(\mathbf{x})\, d\mathbf{x} = |\Omega_i|, \quad \forall i \tag{$**$}$$

$$H\left(\sum q_i p_i\right) = H(q_0) \tag{$***$}$$

$$M\left(\sum q_i p_i\right) = M(q_0) \tag{$****$}$$

This is a particular case of (VP) for q_0 taking only n distinct values.

5. RESOLUTION OF (VP) THE EQUATION OF GIBBS STATES

To proceed further in the determination of the equilibrium set, we have to solve (VP). First, we write down the equation of Gibbs states, which is the equation satisfied by the critical points of the functional I_π on the set $\mathscr{E}$.

Let us assume that ν^* is a solution of (VP) such that $I_\pi(\nu^*) < +\infty$. Then $\nu^* = \rho^*(\mathbf{x}, z)\, \pi$, where $\rho^* \in L^1(\pi)$. Furthermore, we shall assume that

$0 < c \leqslant \rho^* \leqslant C$, π-almost everywhere. Now, applying the Lagrange multipliers rule and performing the same computations as in ref. 25 give

$$\rho^*(\mathbf{x}, z) = \frac{\exp[-\alpha(z) - \beta z \psi^*(\mathbf{x}) - \gamma z y]}{Z(\psi^*(\mathbf{x}), y)}$$

where ψ^* is the stream function associated to the potential vorticity $\bar{v}^*$; β, γ are the Lagrange multipliers of the constraints H, M; $\alpha(z)$ is a continuous function associated to the set of constraints $\int_\Omega v_\mathbf{x}\, d\mathbf{x} = \pi_{q_0}$; and $Z(\psi, y) = \int \exp[-\alpha(z) - \beta z \psi - \gamma z y]\, d\pi_0(z)$.

Thus the stream function ψ^* must satisfy the following equation of Gibbs states:

$$\text{(GSE)} \qquad \left\{ \begin{array}{l} -\Delta\psi + c^2(\psi - \bar{\psi}) = h - \dfrac{1}{\beta}\dfrac{\partial}{\partial\psi} \operatorname{Log} Z \\ \psi = 0 \qquad \text{for} \quad y = \pm m, \quad 2\pi\text{-periodic in } x \end{array} \right\}$$

We have the following existence-uniqueness result for (GSE); where H_0^1 denotes the Sobolev space of the square-integrable functions on the cylinder $\mathscr{C}$, whose first derivatives are also square-integrable and with zero boundary values at $y = \pm m$. The space H_0^1 is endowed with the Hilbert norm $(\int_\Omega (\nabla f)^2\, d\mathbf{x})^{1/2}$. The dual space of H_0^1 identifies in a standard way with the Sobolev space H^{-1} of distributions on the cylinder.

Proposition 5.1. Equation (GSE) always has a solution in the space H_0^1. The solution is unique if $-\beta \max\{z^2 \mid z \in \operatorname{supp} \pi_0\} < \lambda_1$, where λ_1 is the first eigenvalue of the operator $-\Delta$ on the cylinder (associated to the Dirichlet boundary value condition).

Proof. Let us consider the operator $A\psi = -\Delta\psi + c^2(\psi - \bar{\psi})$ from H_0^1 into H^{-1}, One easily checks that A is self-adjoint and coercive:

$$\begin{aligned} \langle A\psi, \psi \rangle &= \int_\Omega (\nabla\psi)^2\, d\mathbf{x} + c^2 \int_\Omega \psi^2\, d\mathbf{x} - \frac{c^2}{|\Omega|}\left(\int_\Omega \psi\, d\mathbf{x}\right)^2 \\ &\geqslant \int_\Omega (\nabla\psi)^2\, d\mathbf{x} \qquad \text{by Cauchy–Schwarz inequality} \end{aligned}$$

Then it follows by the Lax–Milgram lemma that A^{-1} is a continuous operator from H^{-1} into H_0^1, and thus a compact one from $L^2(\mathscr{C})$ into H_0^1.

Now the existence of a solution for the nonlinear equation (GSE) follows by a classical argument, using Schauder's fixed-point theorem, since $(1/Z)(\partial Z/\partial\psi)(\psi, y)$ is a continuous and bounded function.

Let us prove uniqueness. Notice first that any solution of (GSE) is a critical point of the functional

$$\mathscr{J}(\psi)=\frac{1}{2}\int_\Omega (\nabla\psi)^2\,d\mathbf{x}+\frac{c^2}{2}\int_\Omega \psi^2\,d\mathbf{x}-\frac{c^2}{2\,|\Omega|}\left(\int_\Omega \psi\,d\mathbf{x}\right)^2$$

$$+\frac{1}{\beta}\int_\Omega \operatorname{Log} Z(\psi(\mathbf{x}),y)\,d\mathbf{x}-\int_\Omega h\psi\,d\mathbf{x}$$

defined for $\psi\in H_0^1$.

We show now that under the hypothesis of the proposition, the functional $\mathscr{J}$ is strictly convex on the space H_0^1. We calculate the second variation of $\mathscr{J}$:

$$\delta^2\mathscr{J}(\psi)=\frac{1}{2}\int_\Omega (\nabla\delta\psi)^2\,d\mathbf{x}+\frac{c^2}{2}\int_\Omega \delta\psi^2\,d\mathbf{x}$$

$$-\frac{c^2}{2\,|\Omega|}\left(\int_\Omega \delta\psi\,d\mathbf{x}\right)^2+\frac{1}{2\beta}\int_\Omega \frac{\partial^2}{\partial\psi^2}\operatorname{Log} Z(\delta\psi)^2\,d\mathbf{x}$$

Let us denote

$$\zeta(\xi)=\int \exp[-\alpha(z)+z\xi]\,d\pi_0(z)$$

We have

$$\frac{d^2}{d\xi^2}\operatorname{Log}\zeta(\xi)=\frac{\zeta\zeta''-(\zeta')^2}{\zeta^2}$$

and Cauchy–Schwarz inequality gives

$$\left[\int z\exp\left(\frac{-\alpha(z)+z\xi}{2}\right)\exp\left(\frac{-\alpha(z)+z\xi}{2}\right)d\pi_0(z)\right]^2$$

$$\leqslant\int \exp[-\alpha(z)+z\xi]\,d\pi_0(z)\int z^2\exp[-\alpha(z)+z\xi]\,d\pi_0(z)$$

from which $(d^2/d\xi^2)\operatorname{Log}\zeta(\xi)\geqslant 0$.

We also have

$$\frac{d^2}{d\xi^2}\operatorname{Log}\zeta(\xi)\leqslant\frac{\zeta''}{\zeta}\leqslant\max\{z^2\mid z\in\operatorname{supp}\pi_0\}$$

We deduce that for $\beta \geqslant 0$

$$\delta^2 \mathscr{J} \geqslant \tfrac{1}{2} \int_\Omega (\nabla \delta\psi)^2 \, d\mathbf{x}$$

and for $\beta < 0$, using the classical inequality

$$\int_\Omega (\nabla \delta\psi)^2 \, d\mathbf{x} \geqslant \lambda_1 \int_\Omega (\delta\psi)^2 \, d\mathbf{x}$$

we get

$$\delta^2 \mathscr{J} \geqslant \tfrac{1}{2}(\lambda_1 + \beta \max\{z^2 \mid z \in \operatorname{supp} \pi_0\}) \int_\Omega (\delta\psi)^2 \, d\mathbf{x}$$

We see that the hypothesis on β implies the strict convexity of $\mathscr{J}$ and thus the uniqueness of the solution. ■

The link between (GSE) and the variational problem (VP) is given by the following.

Proposition 5.2. Suppose given a continuous function $\alpha(z)$ and real numbers γ, β (satisfying the condition of Proposition 5.1). Let $\psi^{\alpha,\beta,\gamma}$ be the unique solution of (GSE), and $\rho^{\alpha,\beta,\gamma}$ the corresponding density function:

$$\rho^{\alpha,\beta,\gamma}(\mathbf{x}, z) = \frac{1}{Z} \exp[-\alpha(\mathbf{z}) - \beta z \psi^{\alpha,\beta,\gamma}(\mathbf{x}) - \gamma z y], \qquad \nu^{\alpha,\beta,\gamma} = \rho^{\alpha,\beta,\gamma} \pi$$

Then $\nu^{\alpha,\beta,\gamma}$ is the unique solution of the variational problem

$$I_\pi(\nu^{\alpha,\beta,\gamma}) = \min_{\nu \in \mathscr{E}^{\alpha,\beta,\gamma}} I_\pi(\nu)$$

where

$$\mathscr{E}^{\alpha,\beta,\gamma} = \left\{ \nu \in \mathscr{M} \;\middle|\; \int_\Omega \nu_\mathbf{x} \, d\mathbf{x} = \int_\Omega \nu_\mathbf{x}^{\alpha,\beta,\gamma} \, d\mathbf{x}, \right.$$
$$\left. H(\bar{\nu}) = H(\bar{\nu}^{\alpha,\beta,\gamma}) \text{ and } M(\bar{\nu}) = M(\bar{\nu}^{\alpha,\beta,\gamma}) \right\}$$

Proof. See ref. 25, Proposition 7. ■

6. EQUILIBRIUM STATES AND SOLITARY WAVES

In what follows, we shall suppose that q_0 is a patch of potential vorticity of level a surrounded by zero. This leads to some simplifications in the formulas. The discussion of the general case would be similar.

In this particular case, to get the equilibrium set corresponding to q_0, one has to minimize the functional (S is the entropy)

$$-S(p)=\int_\Omega p \operatorname{Log} p+(1-p)\operatorname{Log}(1-p)\,d\mathbf{x}$$

[$p(\mathbf{x})$ is the probability of finding the level a at $\mathbf{x}$] with the constraints

$$\Gamma(ap)=\int_\Omega ap(\mathbf{x})\,d\mathbf{x}=\Gamma(q_0) \qquad (*)$$

$$H(ap)=H(q_0) \qquad (**)$$

$$M(ap)=M(q_0) \qquad (***)$$

The corresponding (GSE) is now

$$\text{(GSE1)} \qquad \begin{cases} -\Delta\psi+c^2(\psi-\bar{\psi})=h(y)+a\dfrac{\exp(-\alpha-\beta a\psi-\gamma ay)}{1+\exp(-\alpha-\beta a\psi-\gamma ay)} \\ \psi=0 \qquad \text{for} \quad y=\pm m, \quad 2\pi\text{-periodic in } x \end{cases}$$

where α is the Lagrange multiplier of the constraint Γ.

Notice first that we can always find solutions of (GSE1) which are only y-dependent (consider $\psi(y)$ and solve the corresponding boundary value problem on $[-m, m]$). For such solutions we have $\nabla\psi \wedge \nabla q=0$, so that they are stationary solutions of QG equations.

But we know that for β lower than a critical value the solution of (GSE1) is no longer unique: a bifurcation occurs with a breaking of the x invariance.[30-32] These x-dependent solutions no longer satisfy the condition $\nabla\psi \wedge \nabla q=0$: they are not stationary solutions of QG equations. Nevertheless, since any such solution satisfies an equation of the form

$$\begin{cases} -\Delta\psi+c^2(\psi-\bar{\psi})=h(y)+f(\psi-c_0 y) \\ \psi=0 \qquad \text{for} \quad y=\pm m, \qquad 2\pi\text{-periodic in } x \end{cases}$$

where $c_0=-\gamma/\beta$, we deduce that $\psi(x-c_0 t, y)$ is a solitary wave traveling at speed c_0 along the x axis (see Appendix C). Thus the equilibrium set is composed of solitary waves.

Let us focus now on what occurs just after the breaking of the x invariance (first bifurcation[30-32]). Then the equilibrium set is obtained by translating along the x axis the potential vorticity q^* associated to any x-dependent bifurcated solution ψ^* of (GSE).

Now we can suggest the following scenario. For t large enough, for example, at time T, the solution $q(T, x, y)$ of (QG) becomes close (for the weak L^2 topology) to an element $q^*(x + x_0, y)$ of the equilibrium set; then for $t \geqslant T$, $q(t, x, y)$ will travel some time close to the solitary wave $q^*(x + (\gamma/\beta)(t - T) + x_0, y)$. If we do again the same reasoning later, we shall get the same solitary wave with only a possible change in the phase x_0.

We remark that the speed of the wave $-\gamma/\beta$ can be determined from the constants of the motion associated to q_0. Indeed, let us denote

$$S^*(\Gamma, H, M) = \max_{\substack{\Gamma(ap) = \Gamma, \\ H(ap) = H, \\ M(ap) = M}} S(p)$$

Since $\delta S = \alpha\, \delta\Gamma + \beta\, \delta H + \gamma\, \delta M$, we have

$$\alpha = \frac{\partial S^*}{\partial \Gamma}, \qquad \beta = \frac{\partial S^*}{\partial H}, \qquad \gamma = \frac{\partial S^*}{\partial M}$$

7. COMMENTS

We refer to refs. 30 and 31 for a detailed discussion, at a physical level, of the consequences of this statistical model. Let us only make here some short comments.

This statistical model is based on the assumption that the organization of the atmospheric flow is purely inertial. That is, the energy exchanges are assumed to be too slow to control the organization of the system at the advective time scale. Strong vorticity could be produced by the action of Coriolis force on rising thermal plumes, which would explain the predominance of anticyclonic spots on the giant planets.

(GSE1) corresponds to the very particular case of an initial datum which is a patch of uniform potential vorticity. The discussion of the general case is similar. Of course the structure of the corresponding equilibrium state (solitary wave) depends on the choice of the initial datum (in fact it only depends on the values of the constants of the motion). This is a complex issue which will be investigated by means of numerical computing in ref. 31.

It was shown in ref. 30 that for accurate values of the parameters an equation like (GSE1) can have a solitary wave solution with a unique vortex. Moreover, the effect of the topography function $\psi_d(y)$ was displayed.

It was shown that a reasonable choice of this topography give a vorticity field which is very close to the one deduced from the observational data from space probes.

It is well known that the spot is almost steady in the rotating frame of reference (in fact, it has only a slow drift of about 1m/sec with respect to the magnetic field of the planet). We can explain this fact: if we suppose that h has the symmetry $h(-y) = h(y)$, then we can show that if $M(q_0) = 0$, then necessarily $\gamma = 0$.[30, 31]

APPENDIX A. YOUNG MEASURES

Let X, Y denote two locally compact separable and metrizable topological spaces. Let us suppose that a positive Borel measure dx (of finite total mass) is given on X. Let us recall that Young measures[35] are a natural way to generalize the notion of measurable mapping from X to Y: at any point $x \in X$, we no longer have a well-determined value, but only some probability distribution on Y. In other words, a Young measure ν is a measurable mapping $x \to \nu_x$ from X to the set $M_1(Y)$ of the Borel probability measures on Y endowed with the narrow topology. Clearly, ν defines a positive Borel measure on $X \times Y$ (which we will also denote by ν) by

$$\langle \nu, f \rangle = \int_X \langle \nu_x, f(x, \cdot) \rangle \, dx$$

for every real function $f(x, y)$, continuous and compactly supported on $X \times Y$ ($f \in C_C(X \times Y)$). Moreover, for $f(x) \in C_c(X)$, we have

$$\langle \nu, f \rangle = \int_X f(x) \, dx$$

that is, the projection of ν on X is dx.

It is well known[11] that this property gives an equivalent definition of Young measures. That is, for any positive Borel measure ν on $X \times Y$ whose projection on X is dx, there is a measurable mapping $x \to \nu_x$ such that the above formula holds. The mapping $x \to \nu_x$ is unique up to the dx-almost everywhere equality.

To any measurable mapping $f: X \to Y$ we associate the Young measure $\delta_f: x \to \delta_{f(x)}$, Dirac mass at $f(x)$.

We shall denote by $\mathscr{M}$ the convex set of Young measures on $X \times Y$, and we recall some useful properties: $\mathscr{M}$ is closed in the space $M_b(X \times Y)$ of all bounded Radon measures on $X \times Y$ (with the narrow topology), the narrow topology is equal on $\mathcal{M}$ to the vague topology (weak topology associated to the continuous compactly supported functions), and it is metrizable. Furthermore, if Y is compact, then $\mathcal{M}$ is compact.

APPENDIX B. LARGE DEVIATIONS AND BALDI'S THEOREM

Baldi's theorem gives general conditions under which a family of probability measures on a locally convex topological vector space has the large-deviation property.

A1. The Large–Deviation Property

Let E be a locally convex Hausdorff topological vector space. We consider a family μ_h, $h>0$, of Borel probability measures on E.

We will say (see, for example, Varadhan[33] or Ellis[6]) that the family μ_h has the large-deviation property with constants $\lambda(h)$ and rate function L iff:

(i) $\lambda(h)>0$ and $\lim_{h\to+\infty}\lambda(h)=+\infty$.

(ii) $L: E\to[0,+\infty]$ is a lower semicontinuous functional on E (not identical to $+\infty$). Moreover, L is inf-compact, that is: the set $\{\nu \mid L(\nu)\leqslant b\}$ is compact for all real numbers b.

(iii) For every Borel subset A of E, we have

$$-\Lambda(\mathring{A})\leqslant \liminf_{h\to\infty}\frac{1}{\lambda(h)}\operatorname{Log}\mu_h(A)$$

and

$$\limsup_{h\to\infty}\frac{1}{\lambda(h)}\operatorname{Log}\mu_h(A)\leqslant -\Lambda(\bar{A})$$

where $\Lambda(A)=\inf_{\nu\in A}L(\nu)$.

The functional L is also usually called the information functional, and $-L$ the entropy functional.

Let E' be the topological dual of E, endowed with the weak-star topology $\sigma(E',E)$. For a Borel probability measure μ on E, we define its Laplace transform:

$$\hat{\mu}(\varphi)=\int_E \exp(\langle\varphi,\nu\rangle)\,d\mu(\nu)\qquad \text{for}\quad \varphi\in E'$$

As is well known, $\hat{\mu}$ is a convex, lower semicontinuous, and proper functional on E'. The same is true for the functional $\operatorname{Log}\hat{\mu}(\varphi)$.

A2. Baldi's Theorem

Let μ_h be a family of Borel probability measures on E, satisfying the following assumptions:

1. There is a function $\lambda(h)$ as in (i) such that

$$\lim_{h \to \infty} \frac{1}{\lambda(h)} \operatorname{Log} \hat{\mu}_h(\lambda(h)\, \varphi) = F(\varphi)$$

where F is a convex, lower semicontinuous, and proper functional on E' which is finite on a neighborhood of the origin.

2. Compacity assumption: For every $R > 0$, there is a compact set $K_R \subset E$ such that

$$\limsup_{h \to +\infty} \frac{1}{\lambda(h)} \operatorname{Log} \mu_h(K_R^c) \leqslant -R$$

Let us denote by L the Young–Fenchel transform of F, that is,

$$L(v) = \sup_{\varphi \in E'} (\langle \varphi, v \rangle - F(\varphi)) \qquad \text{for} \quad v \in E$$

L is a convex, lower semicontinuous, and proper functional on E. We shall suppose that L satisfies the following condition.

3. For every v such that $L(v) < +\infty$, for every open set O containing v and every $\varepsilon > 0$, there is $v_1 \in O$ such that $L(v_1) \leqslant L(v) + \varepsilon$ and L is strictly convex at v_1, that is, $\exists \varphi \in \partial L(v_1)$ such that

$$L(v') > L(v_1) + \langle \varphi, v' - v_1 \rangle \qquad \text{for all} \quad v' \neq v_1$$

Then Baldi's theorem asserts that under the hypotheses 1–3 the family μ_h has the large-deviation property with constant $\lambda(h)$ and rate function L.

A3. Comments

1. L is strictly convex at v if, for example, $\partial L(v)$ is nonempty and

$$L(tv + (1-t)\, v') < tL(v) + (1-t)\, L(v')$$

for all $0 < t < 1$, $v' \in \operatorname{dom} L$, $v' \neq v$.

2. Hypothesis 3 is in fact weaker than the strict-convexity assumption made by Baldi, so that the above statement is slightly different from Baldi's original one[(2)]; for a proof, see Michel.[(15)]

APPENDIX C. SOLITARY WAVE SOLUTIONS OF (QG)

We seek traveling solutions of (QG) of the form

$$\psi(t, x, y) = \Psi(x - c_0 t, y)$$
$$q(t, x, y) = Q(x - c_0 t, y)$$

Then the functions $\Psi(\zeta, y)$, $Q(\zeta, y)$ must satisfy

$$\begin{Bmatrix} -\Delta\Psi + c^2(\Psi - \bar{\Psi}) = h(y) + Q \\ \nabla(\Psi - c_0 y) \wedge \nabla Q = 0 \end{Bmatrix}$$

We see at once that we can get many solitary waves: take any function f, $Q = f(\Psi - c_0 y)$, and solve the nonlinear elliptic equation

$$\begin{Bmatrix} -\Delta\Psi + c^2(\Psi - \bar{\Psi}) = h(y) + f(\Psi - c_0 y) \\ \Psi = 0 \quad \text{for} \quad y = \pm m, \quad 2\pi\text{-periodic in } x \end{Bmatrix}$$

For any continuous bounded function f this equation always has solutions (apply Schauder's fixed-point theorem). Notice that we can always find solutions which are only y-dependent [seek solutions of the form $\psi(y)$ and apply the same argument]; of course such solutions are stationary. But we know that an equation as above can also have bifurcated solutions[30-32] which are x-dependent: these are the genuine solitary waves.

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REFERENCES

1. S. V. Antipov, M. V. Nezlin, E. N. Snezhkin, and A. S. Trubnikov, *Nature* **323**:238–240 (1986).
2. P. Baldi, Large deviations and stochastic homogenization, *Ann. Mat. Pura Appl.* **4**(151):161–177 (1988).
3. E. Caglioti, P. L. Lions, C. Marchioro, and M. Pulvirenti, A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description, *Commun. Math. Phys.* **143**:501–525 (1992).
4. A. J. Chorin, *Lectures on Turbulence Theory* (Publish or Perish, 1975).
5. T. E. Dowling and A. P. Ingersol, *J. Atmos. Sci.* **46**:3256–3278 (1989).
6. R. S. Ellis, *Entropy, Large Deviations and Statistical Mechanics* (Springer-Verlag, 1985).
7. G. L. Eyink and H. Spohn, Negative temperature states and large-scale long-lived vortices in two-dimensional turbulence, *J. Stat. Phys.* **70**(3/4):833–887 (1993).
8. E. J. Hopfinger, Turbulence and vortices in rotating fluids, in *Proceedings of the XVIIth International Congress of Theoretical and Applied Mechanics* (North-Holland, 1989).

9. A. P. Ingersol, Atmospheric dynamics of the outer planets, *Science* **248**:308 (1990).
10. A. P. Ingersol and P. G. Cuong, *J. Atmos. Sci.* **38**:2067–2076 (1981).
11. M. Jirina, On regular conditional probabilities, *Czech. Math. J.* **9**(3):445 (1959).
12. M. K. H. Kiessling, *Commun. Pure Appl. Math.* **46**:27–56 (1993).
13. P. S. Marcus, *Nature* **331**:693–696 (1988).
14. T. Maxworthy and L. G. Redekopp, *Icarus* **29**:261–271 (1976).
15. J. Michel, Thesis, Université Lyon 1 (1993).
16. J. Michel and R. Robert, Large deviations for Young measures and statistical mechanics of infinite dimensional dynamical systems with conservation law, *Commun. Math. Phys.* **159**:195–215 (1994)
17. J. Miller, Statistical mechanics of Euler equations in two dimensions, *Phys. Rev. Lett.* **65**(17):2137–2140 (1990).
18. J. Miller, P. B. Weichman, and M. C. Cross, Statistical mechanics, Euler's equation, and Jupiter's Red Spot, *Phys. Rev. A* **45**:2328–2359 (1992).
19. D. Montgomery and G. Joyce, Statistical mechanics of negative temperature states, *Phys. Fluids* **17**:1139–1145 (1974).
20. P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer-Verlag, 1986).
21. L. Onsager, Statistical hydrodynamics, *Nuovo Cimento* (Suppl. 6) **1949**:279.
22. V. I. Petiashvili, *Sov. Astron. Lett.* **9**:137–138 (1983).
23. J. Pedlosky, *Geophysical Fluid Dynamics*, 2nd ed. (Springer-Verlag, 1987).
24. Y. B. Pointin and T. S. Lundgren, Statistical mechanics of two dimensional vortices in a bounded container, *Phys. Fluids* **10**:1459–70 (1976).
25. R. Robert, A maximum entropy principle for two-dimensional Euler equations, *J. Stat. Phys.* **65**(3/4): 531–553 (1991).
26. R. Robert and J. Sommeria, Statistical equilibrium states for two-dimensional flows, *J. Fluid Mech.* **229**:291–310 (1991).
27. N. N. Romanova and V. Y. Tseytlin, On quasigeostrophic motions in barotropic and baroclinic fluids, *Izv. Acad. Sci. USSR Atmos. Ocean. Phys.* **20**(2):115–124 (1984).
28. N. N. Romanova and V. Y. Tseytlin, Solitary Rossby waves in a weakly stratified medium, *Izv. Acad. Sci. USSR Atmos. Ocean. Phys.* **21**(8):627–630 (1985).
29. J. Sommeria, S. D. Meyers, and H. L. Swinney, Laboratory simulation of Jupiter's Great Red Spot, *Nature* **331**:1 (1988).
30. J. Sommeria, C. Nore, T. Dumont, and R. Robert, Théorie statistique de la tache rouge de Jupiter, *C. R. Acad. Sci. Paris* **312**(Ser. II):999–1005 (1991).
31. J. Sommeria, R. Robert, and T. Dumont, To appear.
32. J. Sommeria, C. Staquet, and R. Robert, Final equilibrium state of a two-dimensional shear layer, *J. Fluid Mech.* **233**:661–689 (1991).
33. S. R. S. Varadhan, Large deviations and applications, in *Ecole d'été de probabilités de Saint-Flour XV–XVII* (1985–87).
34. V. I. Youdovitch, Non stationary flow of an ideal incompressible liquid, *Zh. Vych. Mat.* **3**:1032–1066 (1963).
35. L. C. Young, Generalized surfaces in the calculus of variations, *Ann. Math.* **43**:84–103 (1942).